The correct Heim gravity differential equation.

It seems clear that the Heim original gravity equation is wrong, because of wrong interpretation of field mass. So the question is, what would the correct semi classical differential equation be, if you replace the wrong field mass interpretation with the correct one, taken the other assumptions from Heim to be valid?

The two interpretations are:

Heim: $\varphi = -\frac{G \cdot M_g(r)}{r}$

(Not compatible with Poisson equation $\nabla^2\varphi = 4\pi G\sigma$,$\sigma =$ mass density)

Correct: $\nabla\varphi = \frac{d\varphi}{dr} = \frac{G \cdot M_g(r)}{r^2}$ (Compatible with Poisson equation)

$M_g(r) =$ Gravitational mass. (Point mass + field mass)

According to Heim "Elementarstrukturen der Materie, vol1" page 84:

$$M \cdot c^2 + \frac{\alpha}{2} \cdot \int (\nabla\varphi)^2 dV - \frac{c^2}{4\pi\rho^2}\int \frac{dM}{dr} dV = const. \quad \text{(Heim1)}$$

This is an equation for energy conservation. The first term is the energy of inertial mass, the second the energy stored in the gravitational field, and the third an term , which is rather obscure. However, as it turns out later, this term is generated by an assumption that gravity is caused by graviton particles with rest mass greater than zero. If we take the derivative of the energy equation with respect to r we have:

$$\frac{dM}{dr} + \frac{\alpha}{2c^2} \cdot (\nabla\varphi)^2 \cdot 4\pi r^2 - \frac{r^2}{\rho^2} \cdot \frac{dM}{dr} = 0$$

Or:

$$\frac{dM}{dr} = -\frac{\frac{\alpha}{2c^2} \cdot 4\pi r^2 \cdot (\nabla\varphi)^2}{1 - \frac{r^2}{\rho^2}} \quad (1)$$

According to Heim $\alpha = \frac{3}{64\pi G}$.For it to be consistent with general relativity it should be:

$\alpha = \frac{1}{\pi G}$.

(The Heim derivation of this coefficient is dependent on his interpretation of field mass)

Now, according to Heim (page 83): $\varphi = -\frac{G \cdot M_g(r)}{r} = -\frac{G \cdot M(r) \cdot \left(1 - \frac{r}{\rho}\right)^2}{r}$

This looks strange! Have we not all learned that inertial and gravitational mass always take the same value? In fact, this is not always true. If you look at the relativistic Poisson equation for weak fields $\nabla^2 \varphi = 4\pi G(\sigma + 3p/c^2), \sigma$ =inertial mass density and p is pressure, it is clear that the *gravitational* mass density = $\sigma_g = \sigma + \frac{3p}{c^2}$.

The formula shows, that gravitational and inertial mass only take the same value if you can neglect the pressure term. Heim deduced his correction factor from an assumption that there is a maximum range of ordinary gravity. (r= ρ).

Now what should the correct expression be?

$\frac{d\varphi}{dr} = \frac{G \cdot M(r) \cdot f(r)}{r^2}$ where f is some unknown correction factor.

If you solve original Heim equation without the field energy term (Which is causing the problems with Heim original solution):

$\varphi = -\frac{GM}{r}\left(1 - \frac{r}{\rho}\right)^2$ where M is the fixed point mass.

If we take the derivative of this expression we have:

$$\frac{d\varphi}{dr} = -\frac{G \cdot M \cdot \left(1 - \frac{r^2}{\rho^2}\right)}{r^2}$$

This leads us to the plausible assumption that $f(r) = 1 - \frac{r^2}{\rho^2}$

Hence: $$\frac{d\varphi}{dr} = \nabla\varphi = \frac{G \cdot M(r) \cdot \left(1 - \frac{r^2}{\rho^2}\right)}{r^2} \quad (2)$$

Now, let us substitute this expression in equation (1):

$$\frac{dM(r)}{dr} = -\frac{2}{c^2} \cdot \frac{G \cdot M(r)^2}{r^2} \cdot \left(1 - \frac{r^2}{\rho^2}\right)$$

This is a rather simple separable differential equation.

If we integrate:

$$\int_{M_0}^{M}\frac{dM}{M^2}=-\frac{2G}{c^2}\cdot\int_{\rho}^{r}\left(\frac{1}{r^2}-\frac{1}{\rho^2}\right)dr=-\frac{2G}{rc^2}\cdot\left(1-\frac{r}{\rho}\right)^2$$

Here M_0 is the point mass of the system as seen from where gravity field is zero. (Gravity is zero when r = ρ). If we solve for M, we have:

$$M(r)=\frac{M_0}{1-\frac{2GM_0}{rc^2}\cdot\left(1-\frac{r}{\rho}\right)^2}$$ (Inertial mass as seen from r= ρ) (3)

$$M_g(r)=\frac{M_0\cdot\left(1-\frac{r^2}{\rho^2}\right)}{1-\frac{2GM_0}{rc^2}\cdot\left(1-\frac{r}{\rho}\right)^2}$$ (Gravitational mass as seen from r= ρ)

$$\frac{d\varphi}{dr}=\nabla\varphi=\frac{\frac{GM_0}{r^2}\cdot\left(1-\frac{r^2}{\rho^2}\right)}{1-\frac{2GM_0}{rc^2}\cdot\left(1-\frac{r}{\rho}\right)^2}$$ (Gravitational field as seen from r= ρ)

$$\varphi=\frac{c^2}{2}\cdot\ln\left(1-\frac{2GM_0}{rc^2}\cdot\left(1-\frac{r}{\rho}\right)^2\right)$$ (Gravitational potential field as seen from r= ρ)

In general relativity $\varphi_{local}=-c^2\left(1-e^{\frac{\varphi}{c^2}}\right)$, where φ_{local} is the field seen from a stationary observer at the point where you measure the field. If you calculate φ_{local} with the above solutions, you get the exact solutions to the GRT field equations with the field source:

$$\sigma(r)=\frac{M_0}{2\pi\rho^2}\cdot\frac{1}{r}$$

$p_r=-\sigma(r)\cdot c^2\Rightarrow$ Dark energy!

$$p_\perp=\frac{p_r}{2}$$

(See "Heim gravity differential equation, Börje Månsson")

This shows that the assumption f(r) = $1-\left(\frac{r}{\rho}\right)^2$ is consistent with general relativity.

Now, what would the correct differential equation be involving derivatives of φ? (The solution of which we already know). Equation (2) gives:

$$M(r) = \frac{r^2 \cdot \frac{d\varphi}{dr}}{G \cdot \left(1 - \frac{r^2}{\rho^2}\right)}$$. Substitute this into (1):

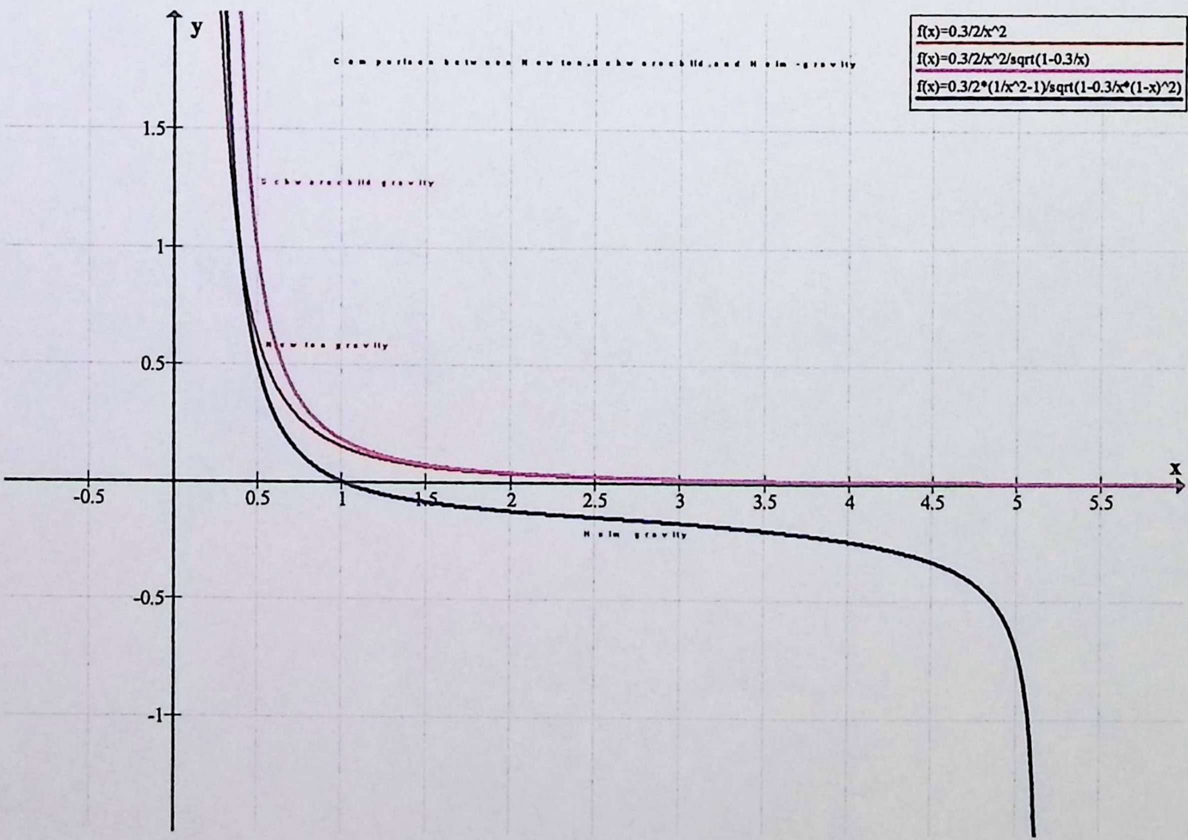
$$\frac{d}{dr}\left(\frac{r^2 \cdot \frac{d\varphi}{dr}}{G \cdot \left(1 - \frac{r^2}{\rho^2}\right)}\right) = -\frac{2 \cdot r^2 \cdot \left(\frac{d\varphi}{dr}\right)^2}{Gc^2 \cdot \left(1 - \frac{r^2}{\rho^2}\right)}$$

If you simplify this equation you get the "correct" Heim equation:

$$\boxed{\frac{d^2\varphi}{dr^2} + \frac{2}{r} \cdot \frac{1}{1 - \frac{r^2}{\rho^2}} \cdot \frac{d\varphi}{dr} + \frac{2}{c^2} \cdot \left(\frac{d\varphi}{dr}\right)^2 = 0}$$

Solution: $\varphi = \frac{c^2}{2} \cdot \ln\left(1 - \frac{2GM_0}{rc^2} \cdot \left(1 - \frac{r}{\rho}\right)^2\right)$, $\quad \frac{d\varphi}{dr} = \frac{\frac{GM_0}{r^2} \cdot \left(1 - \frac{r^2}{\rho^2}\right)}{1 - \frac{2GM_0}{rc^2} \cdot \left(1 - \frac{r}{\rho}\right)^2}$

A graph of the gravity field (acceleration-field) for Newton, Schwarzschild and Heim:



According to Heim $\rho = \dfrac{h^2}{Gm^3}$, where m is the elementary particle masses building up the point mass M. Heim is deriving this formula by assuming that the loss of total inertial mass from $r = r_0$ to $r = \rho$ is the mass of a graviton particle generated by the particle m. (Page 87 Elementarstrukturen der Materie, vol1) The result of this derivation is dependent of which model you have for field mass. As we don't have the same model, we can't expect to get the same result. What will the formula be with the new interpretation?

Let us use the same basic assumption as Heim:

$-\mu_g = m(\rho) - m(r_0)$, where μ_g is the mass of graviton particle. If we use equation (3) we have:

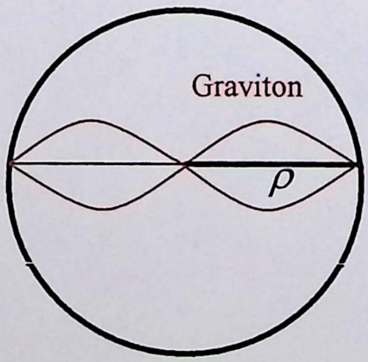
$$-\mu_g = M(\rho) - M(r_0) = m_0 - \frac{m_0}{1 - \frac{2Gm_0}{r_0 c^2} \cdot \left(1 - \frac{r_0}{\rho}\right)^2}$$

, where m_0 is the mass m as measured from $r = \rho$. Here it would be better to use the mass m, measured from $r = r_0$.

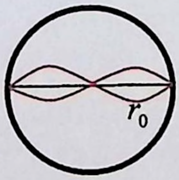
We have $m = \dfrac{m_0}{1 - \frac{2Gm_0}{r_0 c^2} \cdot \left(1 - \frac{r_0}{\rho}\right)^2} \Rightarrow m_0 = \dfrac{m}{1 + \frac{2Gm}{r_0 c^2} \cdot \left(1 - \frac{r_0}{\rho}\right)^2}$ .If we substitute this into the equation above, we get:

$$-\mu_g = \frac{m}{1 + \frac{2Gm}{r_0 c^2} \cdot \left(1 - \frac{r_0}{\rho}\right)^2} - m = -\frac{\frac{2Gm^2}{r_0 c^2} \cdot \left(1 - \frac{r_0}{\rho}\right)^2}{1 + \frac{2Gm}{r_0 c^2} \cdot \left(1 - \frac{r_0}{\rho}\right)^2}. \quad (4)$$

Heim assumes that the Compton wave length for the graviton particle is 2ρ and the Compton wave length for the particle $2r_0$.



Universe



Particle m

The Compton wavelength for a particle is $\lambda = \frac{h}{m \cdot c}$. So we have for the graviton:

$$2\rho = \frac{h}{\mu_g c} = \frac{h}{c} \cdot \frac{1 + \frac{2Gm}{r_0 c^2} \cdot \left(1 - \frac{r_0}{\rho}\right)^2}{\frac{2Gm^2}{r_0 c^2} \cdot \left(1 - \frac{r_0}{\rho}\right)^2} = \frac{h}{mc} + \frac{hcr_0}{2Gm^2} \cdot \frac{1}{\left(1 - \frac{r_0}{\rho}\right)^2}$$

For the particle we have $2r_0 = \frac{h}{mc}$. If we substitute this in the equation above we get:

$$2\rho = 2r_0 + \frac{h^2}{4Gm^3} \cdot \frac{1}{\left(1 - \frac{r_0}{\rho}\right)^2}$$

Or:

$$\boxed{\rho = r_0 + \frac{h^2}{8Gm^3} \cdot \frac{1}{\left(1 - \frac{r_0}{\rho}\right)^2}}$$

This is an cubic algebraic equation in ρ. However we expect $\rho \succ\succ r_0$. Then we have: $\boxed{\rho = \frac{h^2}{8Gm^3}}$ which will hold as long as m is small. We notice that ρ is 8 times smaller than in Heim's original formula. The graviton mass will be:

$$\mu_g = \frac{h}{2\rho c} = \frac{4Gm^3}{hc}$$

Graviton mass generated by a proton:

$$\begin{aligned} G &= 6.67 \cdot 10^{-11} Nm^2/kg^2 \\ h &= 6.63 \cdot 10^{-34} Js \\ c &= 3.0 \cdot 10^8 m/s \\ m &= 1.66 \cdot 10^{-27} kg \end{aligned} \quad \Rightarrow \mu_g = 6.13 \cdot 10^{-66} kg$$

Now, back to the strange term $\frac{c^2}{4\pi\rho^2}\int \frac{dM}{dr} dV$ in the equation (Heim1):

$$M(r) \cdot c^2 + \frac{\alpha}{2} \cdot \int (\nabla\varphi)^2 dV - \frac{c^2}{4\pi\rho^2}\int \frac{dM}{dr} dV = const = M(r_0) \cdot c^2$$

$$\Rightarrow M(r_0) - M(\rho) = \frac{\alpha}{2c^2} \cdot \int_{r_0}^{\rho} (\nabla\varphi)^2 dV - \frac{1}{4\pi\rho^2}\int_{r_0}^{\rho} \frac{dM}{dr} dV$$

According to Heim, this is the mass of the graviton.
The first term, which is still nonzero when $\rho \to \infty$, can be interpreted as the kinetic mass of the graviton, and the other as the **rest mass** of the graviton.
So, In effect Heim gravity is a theory of massive gravitons. In standard theory the graviton rest mass is exactly zero.
How large is the rest mass of the graviton?

$$\mu_{g,0} = -\frac{1}{4\pi\rho^2}\int_{r_0}^{\rho}\frac{dM}{dr}dV = -\frac{1}{\rho^2}\cdot\int_{r_0}^{\rho}\frac{dM}{dr}\cdot r^2 dr = \frac{1}{\rho^2}\cdot\int_{r_0}^{\rho}\frac{\frac{2Gm_0^{\,2}}{c^2}\left(1-\frac{r^2}{\rho^2}\right)dr}{\left(1-\frac{2Gm_0}{rc^2}\left(1-\frac{r}{\rho}\right)^2\right)^2}$$

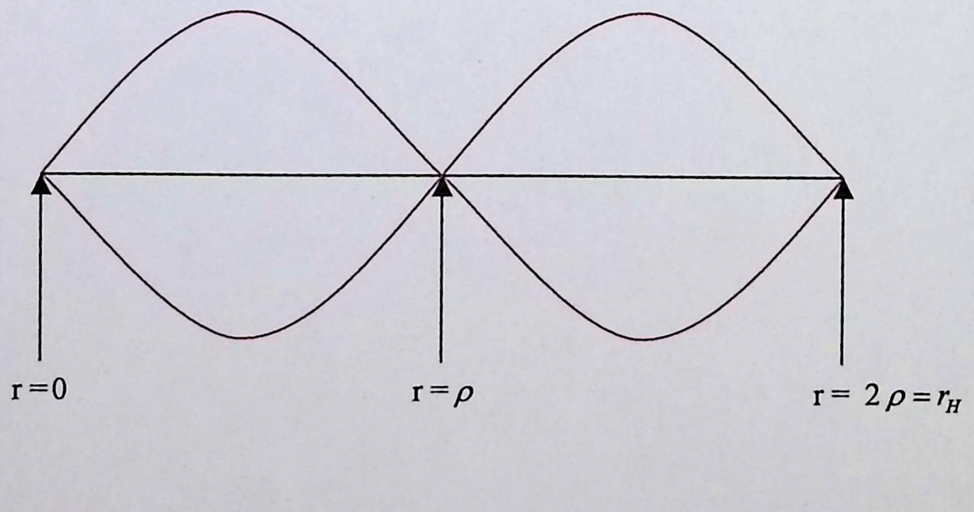
$$= \frac{2Gm_0^{\,2}}{\rho c^2}\cdot\int_{\frac{r_0}{\rho}}^{1}\frac{(1-x^2)dx}{\left(1-\frac{2Gm_0}{\rho c^2}\cdot\frac{(1-x)^2}{x}\right)^2} \approx \frac{2Gm_0^{\,2}}{\rho c^2}\cdot\int_0^1\left(1-x^2\right)dx \quad (\text{If } r_0 >> \frac{2Gm_0}{c^2} \text{ and } \rho >> r_0 .)$$

$$\mu_{g,0} = \frac{2Gm_0^{\,2}}{\rho c^2}\cdot\int_0^1\left(1-x^2\right)dx = \frac{4Gm_0^{\,2}}{3\rho c^2} = \frac{32}{3}\cdot\frac{Gm^2}{\left(1+\frac{2Gm}{r_0c^2}\right)^2 c^2}\cdot\frac{Gm^3}{\hbar^2} \approx \frac{32}{3}\cdot\frac{G^2m^5}{\hbar^2c^2}$$

, where ρ is substituted with $\rho = \dfrac{\hbar^2}{8Gm^3}$.

For a proton the graviton *rest mass* will be
$\mu_{g,0}(proton) = 1{,}5\cdot 10^{-104}\,kg$, which is extremely small, but not zero.

We can interpret the graviton field of entire universe as a standing wave with a node at $r=\rho$. Where is the next node? It is natural to assume that the next node is at the maximum radius of influence, $r = r_H$ the Hubble radius.



If we use the result from "Heim gravity differential equation, Börje Månsson") that Heim gravity (corrected) can be made a solution to GR if every mass M_0 generates a "dark energy field" $\sigma(r)=\frac{M_0}{2\pi\rho^2}\cdot\frac{1}{r}$, which summed up over visible universe generates the cosmological dark energy field, we have if we assume a constant density σ of matter(ordinary matter+ dark matter):

$$\sigma_{dark}=\frac{M(r_H)}{2\pi\rho^2}\cdot\frac{1}{r_H}=\frac{\sigma\cdot\frac{4\pi}{3}\cdot r_H^{\;3}}{2\pi\rho^2\cdot r_H}=\frac{2}{3}\cdot\left(\frac{r_H}{\rho}\right)^2\cdot\sigma$$

, where r_H is the maximum radius of influence. (The Hubble radius)
If we substitute $r_H=2\rho$ we get:

$$\sigma_{dark}=\frac{8}{3}\cdot\sigma$$

$$\Omega_\Lambda=\frac{\sigma_{dark}}{\sigma_{dark}+\sigma}=\frac{\frac{8}{3}}{\frac{8}{3}+1}=\frac{8}{11}\approx 0.727$$

This should be compared to the experimental value:

Ω_Λ (Experimental)$=0.725\pm 0.018$. Close enough!

Now we can calculate the average graviton mass in universe:

$$one\ wavelength=r_H=\frac{h}{\mu_g(universe)\cdot c}\Rightarrow \mu_g(universe)=\frac{h}{r_H\cdot c}$$

with $r_H=1.28\cdot 10^{26}\,m$,we get:

$$\boxed{\mu_g(universe)=1.73\cdot 10^{-68}\,kg}$$

Here

$$r_H=\sqrt{\frac{3c^2}{8\pi G(\sigma+\sigma_\Lambda)}}=\sqrt{\frac{3c^2\cdot\Omega_\Lambda}{8\pi G\sigma_\Lambda}}=\sqrt{\frac{3c^2\cdot\Omega_\Lambda}{8\pi G\cdot\frac{c^2\cdot\Lambda}{8\pi G}}}=\sqrt{\frac{3\cdot\Omega_\Lambda}{\Lambda}}\Rightarrow \mu_g(universe)=\frac{h}{c\cdot\sqrt{\frac{3\cdot\Omega_\Lambda}{\Lambda}}}=\frac{h}{c}\cdot\sqrt{\frac{\Lambda}{3\cdot\Omega_\Lambda}}$$

$$\mu^2{}_g=\frac{h^2}{c^2}\cdot\frac{\Lambda}{3\cdot\Omega_\Lambda}=\frac{1}{3\cdot\frac{8}{11}}\cdot\frac{h^2}{c^2}\cdot\Lambda=\frac{11}{24}\cdot\frac{h^2}{c^2}\cdot\Lambda\approx 0.458\cdot\frac{h^2}{c^2}\cdot\Lambda$$

This should be compared to the result from another theory of gravitons:

$$\mu^2{}_g = \frac{h^2}{c^2}\cdot\frac{2\cdot\Lambda}{3} \approx 0.667\cdot\frac{h^2}{c^2}\cdot\Lambda$$ (Novello Class. Quantum Grav.20 (2003)L67-L73)

very close to the result above.
Mostly, other theories of gravitons generates in the weak field limit a differential equation of PROCA-type:

$$\nabla^2\varphi - \frac{\varphi}{\lambda^2} = 0$$

Here $\lambda =$ Compton wavelength of graviton$= \dfrac{h}{\mu_{g0}\cdot c}$.

In the weak field limit for Heim-gravity we have

$$\frac{d\varphi}{dr} = GM\left(\frac{1}{r^2} - \frac{1}{\rho^2}\right) \Rightarrow \nabla^2\varphi = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\varphi}{dr}\right) = \frac{1}{r^2}\frac{d}{dr}\left(GM\left(1 - \frac{r^2}{\rho^2}\right)\right) = -\frac{2GM}{\rho^2 r}$$

According to weak field Poisson equation

$$4\pi G\sigma_g = \nabla^2\varphi = -\frac{2GM}{\rho^2 r} \Rightarrow \sigma_g = -\frac{M}{2\pi\rho^2 r} = \frac{\varphi_{newton}}{2\pi G\rho^2}$$

, where φ_{newton} is the Newton potential field.
So the graviton is generating a negative gravitational mass field. (Anti gravitating)
The Poisson equation can now be written:

$$\nabla^2\varphi - \frac{2\varphi_{newton}}{\rho^2} = 0$$. Here $2\rho = \lambda$ according to Heim.

$$\nabla^2\varphi - \frac{8\varphi_{newton}}{\lambda^2} = 0$$

Apart from the factor 8 this is identical to the PROCA equation if we approximate $\varphi \approx \varphi_{newton}$. However, there is a difference in the interpretation of Compton wavelength. Other theories of gravitons assumes it to be $\dfrac{h}{\mu_{g0}\cdot c}$, (Or $\dfrac{\hbar}{\mu_{g0}\cdot c}$) while Heim assumes it is $\dfrac{h}{\mu_g\cdot c}$. Other theories use rest mass for the graviton mass, Heim total mass, in the formula for Compton wavelength.

Summery:

If you substitute the original (wrong) field mass model, with the correct one, and use the other ideas from Heim in setting up the differential equation for gravity, you end up with a semi classical differential equation:

$$\boxed{\frac{d^2\varphi}{dr^2}+\frac{2}{r}\cdot\frac{1}{1-\frac{r^2}{\rho^2}}\cdot\frac{d\varphi}{dr}+\frac{2}{c^2}\cdot\left(\frac{d\varphi}{dr}\right)^2=0}$$

The original Heim formula for ρ is replaced with:

$$\boxed{\rho=\frac{h^2}{8Gm^3}}$$

The total mass of a graviton is:

$$\boxed{\mu_g=\frac{4Gm^3}{hc}}$$

The *rest mass* of a graviton is:

$$\boxed{\mu_{g0}=\frac{32}{3}\cdot\frac{G^2m^5}{h^2c^2}}$$

Heim derives his formula for ρ by assuming that the loss of inertial mass of an elementary particle from r=0 to r= ρ is identical to the total mass of the graviton.

Heim's theory is in effect a theory of massive gravitons.

Dark energy is generated by massive gravitons coming from ordinary elementary particles. The new differential equation has the same solution as the GR solution to the field equations with a dark energy mass density source:

$$\sigma(r)=\frac{M_0}{2\pi\rho^2}\cdot\frac{1}{r}$$

$p_r=-\sigma(r)\cdot c^2 \Rightarrow$ Dark energy!

$$p_\perp=\frac{p_r}{2}$$

If you integrate this over the visible universe, you come up with the theoretical value:

$$\sigma_{dark}=\frac{8}{3}\cdot\sigma$$

where σ is the density of ordinary matter(All but dark energy)

and

$$\boxed{\Omega_\Lambda=\frac{8}{11}\approx 0.727}$$

This value is very close to the experimental value.

There are other theories of massive gravitons. The connection formula between graviton mass and cosmological constant are almost identical in (corrected) Heim and other theories. In the weak field limit, Heim (corrected) theory goes over to a PROCA-like differential equation, just as in other theories of massive gravitons.

/Börje Månsson

The problem with mass in the solution to the the differential equation $\frac{dm}{dr}+a\cdot\left(\frac{m}{r}\right)^2=0$

This equation can be deduced by using the energy principle

$$m(r)c^2 = m(r_0)\ c^2 - \frac{\alpha}{2}\cdot\int_{r_0}^{r}\left(\frac{d\varphi}{dr}\right)^2 dV\ .$$

Here α should be $\frac{1}{\pi G}$, not $\frac{3}{64\pi G}$.

(To be consistent with GR.) Heim derived his α with wrong interpretation of field mass.

The solution to this is $\frac{m(r)}{m(r_0)} = \frac{1}{1+\frac{a\cdot m(r_0)}{r_0}\left(\frac{r-r_0}{r}\right)}$, where $a=\frac{2G}{c^2}$ (Not $\frac{3G}{64c^2}$)

Here $m(r_0)$ is the mass measured from $r=r_0$. The problem with this is that the mass measured is not independent from where you measure it! If you choose another r_0, you get another mass. This is caused by the field energy term

$$-\frac{\alpha}{2}\cdot\int_{r_0}^{r}\left(\frac{d\varphi}{dr}\right)^2 dV\ .$$

The further away , the lesser mass. To measure the mass from r_0 is not a good choice. It is better measure it from infinity, where the field energy cannot diminish it further. If you just put $r_0=\infty$ in the above equation, you have:

$\frac{m(r)}{m} = \frac{1}{1-\frac{a\cdot m}{r}}$. Here m= mass of system measured from infinity. Now, you have no dependence of r_0. However, if another mass is appearing in the neighbourhood, m will be changed as, the gravitational field around m changes.

If the two masses are m_1, m_2 after they come together, m_{10}, m_{20} before they come together ,we have:

$$m_1c^2 + m_2c^2 = m_1(r_1)c^2 + m_2(r_2)c^2 - \frac{\alpha}{2}\int_{spaceoutsider1,r2}\left(\frac{d\varphi_1}{dr}+\frac{d\varphi_2}{dr}\right)^2 dV$$

$$m_{10}c^2 + m_{20}c^2 = m_{10}(r_1)c^2 + m_{20}(r_2)c^2 - \frac{\alpha}{2}\int_{r_1}^{\infty}\left(\frac{d\varphi_1}{dr}\right)^2 dV - \frac{\alpha}{2}\int_{r_2}^{\infty}\left(\frac{d\varphi_2}{dr}\right)^2 dV$$

$$m_1(r_1)c^2 = m_{10}(r_1)c^2 - \frac{\alpha}{2}\int_{0}^{r_1}\left(\frac{d\varphi_2}{dr}\right)^2 dV \quad , \quad m_2(r_2)c^2 = m_{20}(r_2)c^2 - \frac{\alpha}{2}\int_{0}^{r_2}\left(\frac{d\varphi_1}{dr}\right)^2 dV$$

$$m_1 + m_2 = m_{10} + m_{20} - \frac{\alpha}{2c^2}\int_0^{r_1}\left(\frac{d\varphi_2}{dr}\right)^2 dV - \frac{\alpha}{2c^2}\int_0^{r_2}\left(\frac{d\varphi_1}{dr}\right)^2 dV - \frac{\alpha}{2c^2}\int_{r_1}^{\infty}\frac{d\varphi_1}{dr}\cdot\frac{d\varphi_2}{dr}dV - \frac{\alpha}{2c^2}\int_{r_2}^{\infty}\frac{d\varphi_1}{dr}\cdot\frac{d\varphi_2}{dr}dV$$

This means that when two masses are brought together some of the total mass is transferred over to field mass. You have to treat the field mass as a third body in the system. So when you consider conservation of momentum, or equivalently that the sum of all interacting gravitational forces should be zero, you should expect $\vec{F}_1 + \vec{F}_2 + \vec{F}_{field} = 0 \Rightarrow \vec{F}_1 + \vec{F}_2 = -\vec{F}_{field}$.
This means, you cannot expect the momentum for the two original masses to be conserved if there is an asymmetric condition. If there is a totally symmetric condition $m_1 = m_2; r_1 = r_2$ the two mixed integrals will vanish, and the two first integrals will just diminish the masses within the two bodies with the same amount.
However, I do not think this means that the centre of mass of the two bodies can self accelerate in a linear manner if you have an asymmetric condition, as it partly drags its gravitational field with it(Just partly because it takes time to inform the outer part of the gravitational field what's going on). Probably there will be a slight wiggle back and forth of the mass centre. (Very slowly)

/Börje Månsson